# Observables have no value:
# a no-go theorem for position and momentum observables

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The Bell-Kochen-Specker contradiction is presented using continuous observables in infinite dimensional Hilbert space. It is shown that the assumption of the *existence* of putative values for position and momentum observables for one single particle is incompatible with quantum mechanics.



## I. INTRODUCTION

One of the central questions in the interpretation of quantum mechanics from a realist perspective is whether the indeterminacies or uncertainties in quantum mechanics are of an *ontological* or *gnoseological* character. They are gnoseological if the observables of the system possess exact values that quantum mechanics is unable to predict and can only provide probability distributions for them. In this case the uncertainty is in our knowledge of the system and not in the system itself and the development of a deterministic theory with hidden variables that are averaged to produce the same statistical predictions of quantum mechanics is wished. The indeterminacies are ontological if the observables do not assume exact values but instead they are diffuse by nature and the indeterminacies are in nature and not in our knowledge. In this case quantum mechanics can be considered to be a complete theory and not the statistical average of a better theory. The Einstein-Podolsky-Rosen argument[1] was originally designed in order to prove that quantum mechanics is not complete, although later developments favour an interpretation of the argument where the values of the "elements of physical reality" are nonlocality determined (a special case of contextuality). Indeed, the experimental violations[2] of Bell's inequality[3] have established such nonlocal effects in the valuation of observables.

The existence of definite context independent values for the observables was shown by Bell[4] and by Kochen and Specker[5] to be in conflict with quantum mechanics on logical grounds, that is, in conflict with the geometrical structure of the Hilbert space, more than with the postulates of quantum mechanics. The Kochen Specker theorem is a complicated argument requiring 117 vectors in a three dimensional Hilbert space. A simpler proof was produced by Peres[6] with only 33 vectors and Penrose[7] found a beautiful geometrical representation for these vectors. With another goal, not trying to minimize the number of directions, a proof of the theorem was given[8] involving continuous sets of directions. Analysing spin observables for systems of two and three particles, Mermin[9] presented physical examples of the Bell-Kochen-Specker contradiction.

The original Einstein-Podolsky-Rosen argument involves observables of position and momentum and D. Bohm[10] presented the same argument but in terms of spin observables. Since then, spin observables were preferred for Einstein-

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Podolsky-Rosen and Bell-Kochen-Specker type of arguments. This preference is not only because these involve a finite dimensional Hilbert space with simpler mathematics, but mainly because spin observables are more adequate for real experimental tests. However, spin is an essentially quantum mechanic observable with almost no correlate in classical mechanics and therefore it is interesting to exhibit these arguments also for position and momentum observables in order to emphasize the drastic differences between classical and quantum mechanics. With these observables, a "frame function" was constructed and a rigourous proof of the Bell-Kochen-Specker theorem was provided[11]. Furthermore, illustrations of the Bell-Kochen-Specker contradiction were built involving Sign and Reflection observables[12] and also for unitary operators, functions of position and momentum[13]. In all these cases, the operators involved are not trivial functions of position and momentum and it would be convenient to devise a simple proof. In this work, the incompatibility of quantum mechanics with the assumption of the existence of noncontextual values for position and momentum of just one particle is shown. The proof is very simple (*a posteriori*) and it involves the simplest physical system (one spinless free particle) and therefore it makes the Bell-Kochen-Specker result easier accessible to non experts.

In this work we will investigate the possibility of existence of definite values for position and momentum observables of a single particle. Of course these putative values are not provided by quantum mechanics and it does not matter whether they are deterministic, as in a hidden variable theory, or are random values distributed according to some inherent randomness in nature (zitterbewegung). In an ensemble of systems, we only require that the proposed putative values should be distributed according to the distributions predicted by quantum mechanics.

## II. THE PUTATIVE VALUE

Let us assume that we can assign to any Hilbert space operator $A$ a numerical value $\overline{A}$ called the *putative value*, with the following properties.

- *Completeness*: the set $\{\overline{A}\}$ of all possible putative values is the spectrum of the corresponding operator.
- *Functional consistency:* the putative value preserves functional relations in the sense that for any function $F$ it is $\overline{F(A)} = F(\overline{A})$.
- *Context independence:* the putative value assumed by an operator is independent of the context in which the corresponding observable is placed. Different contexts are defined by different sets of commuting operators.

It can be proved[14] that the functional consistency condition has the important consequence that the putative value for *commuting* operators are additive and multiplicative. That is,

$$[A, B] = 0 \;\rightarrow\; \overline{A+B} = \overline{A} + \overline{B} \;\text{ and }\; \overline{A \cdot B} = \overline{A} \cdot \overline{B}\ . \tag{1}$$

These relations can be generalized to functions that can be expanded as power series: $\overline{F(A,B)} = F(\overline{A}, \overline{B})$, however we will not use the general form. In fact we will only need the additive property. This additive property is not necessarily true for non commuting operators. For instance if $A = J_x$ and $B = J_y$ are two components of angular momentum, then $A + B = \sqrt{2}J_u$ is also a component of angular momentum in a different direction but the spectrum of $\sqrt{2}J_u$ is clearly not equal to the sum of the spectra of $J_x$ and $J_y$.

We will now investigate whether it is possible to assign putative values $\overline{X}$ and $\overline{P}$ to the position and momentum observables $X$ and $P$ of a particle, in a way compatible with quantum mechanics. This compatibility means that the putative values should be distributed according to the probability functions provided by quantum mechanics. Of

course, quantum mechanics can not predict or compute these putative values but we want to know if their *existence* is allowed by quantum mechanics. We just want to see if we can *think* that these values exist. It is also irrelevant whether these values can be calculated by a deterministic hidden variables theory or they are assigned randomly.

Let us assume that the position observable is divided by some length scale $\lambda$ (Compton length, for instance) and momentum is multiplied by $\lambda/\hbar$ making them dimensionless. Therefore their associated values are pure numbers and the addition of position with momentum is not meaningless. Let us consider the operators $X_1, X_2, P_1, P_2$ corresponding to the observables of position and momentum of a particle in a plane and let $\overline{X}_1, \overline{X}_2, \overline{P}_1, \overline{P}_2$ be their putative values. Let us now build several linear combinations of these operators that can be grouped in intersecting subsets of commuting operators. In Fig.1 we see some of these linear combinations and we notice that all operators joined by a straight line commute. Let us consider the two operators $A = X_1 - X_2 + P_1 + P_2$ and $B = X_1 + X_2 + P_1 - P_2$. Since they commute, their putative values are such that

$$\overline{A+B} = \overline{A} + \overline{B} \ . \tag{2}$$

The left hand side of this equation is $\overline{A+B} = \overline{2X_1 + 2P_1}$, and the right hand side is $\overline{A} + \overline{B} = \overline{X_1 - X_2 + P_1 + P_2} + \overline{X_1 + X_2 + P_1 - P_2} = \overline{X_1 - X_2} + \overline{P_1 + P_2} + \overline{X_1 + X_2} + \overline{P_1 - P_2} = \overline{X}_1 - \overline{X}_2 + \overline{P}_1 + \overline{P}_2 + \overline{X}_1 + \overline{X}_2 + \overline{P}_1 - \overline{P}_2 = 2\overline{X}_1 + 2\overline{P}_1$. Therefore the equation above becomes

$$\overline{X_1 + P_1} = \overline{X}_1 + \overline{P}_1 \ . \tag{3}$$

We have used the additive property of the putative values of *commuting* observables and we have shown that, even though $X_1$ and $P_1$ *do not commute*, their putative values are also additive. This is indeed very suspicious and for most experts this would be sufficient reason to deny the existence of the putative values. Anyway we will prove that this result is in contradiction with quantum mechanics, but before doing this, some comments are convenient.

The property of context independence is necessary in the above argument because we assume that the value of, for instance, the operator $X_1 + X_2$ in the upper right corner of Fig.1, is the same when we consider it as a member of the set $\{X_1 - X_2, X_1, X_2, X_1 + X_2\}$ as the value it takes when it is a member of the set $\{X_1 + X_2, B, P_1 - P_2\}$. Without this assumption, Eq.(3) could not be obtained. We could have taken other sets of commuting operators leading to similar results. For instance in the top line of Fig.1 we could take the operators $X_1 + P_2, X_1, P_2, X_1 - P_2$ and the appropriate set in the lower line. Also, choosing the signs properly, instead of an addition we could get a substraction in Eq.(3) or any linear combination of the operators. Considering the projection of position and momentum along two arbitrary orthogonal directions in three dimensional space, would lead us to the conclusion that for any linear combination we have $\overline{\alpha\mathbf{X} + \beta\mathbf{P}} = \alpha\overline{\mathbf{X}} + \beta\overline{\mathbf{P}}$. Notice that in order to obtain these results it is important that the commutator of $X$ and $P$ is a *constant* and a subtle cancellation of the commutator in different directions is made. This cancellation is no longer possible when we use a similar scheme in order to try to prove that $\overline{X^2 + P^2} = \overline{X^2} + \overline{P^2}$. Unfortunately several attempts to prove this failed. If we could prove this, then we would obtain an immediate contradiction because the spectrum of $X^2 + P^2$ is discrete whereas the spectra of $X^2$ and $P^2$ are continuous. An immediate contradiction would also follow if we could prove that $\overline{XP} = \overline{X}\ \overline{P}$ because in one case the spectrum is complex and in the other it is real.

## III. CONTRADICTION WITH QUANTUM MECHANICS

If the putative values for position $X$ and momentum $P$ exist, then they must be such that, for the operator $S = X + P$, we have $\overline{S} = \overline{X+P} = \overline{X} + \overline{P}$. We will now see that this is in contradiction with quantum mechanics.

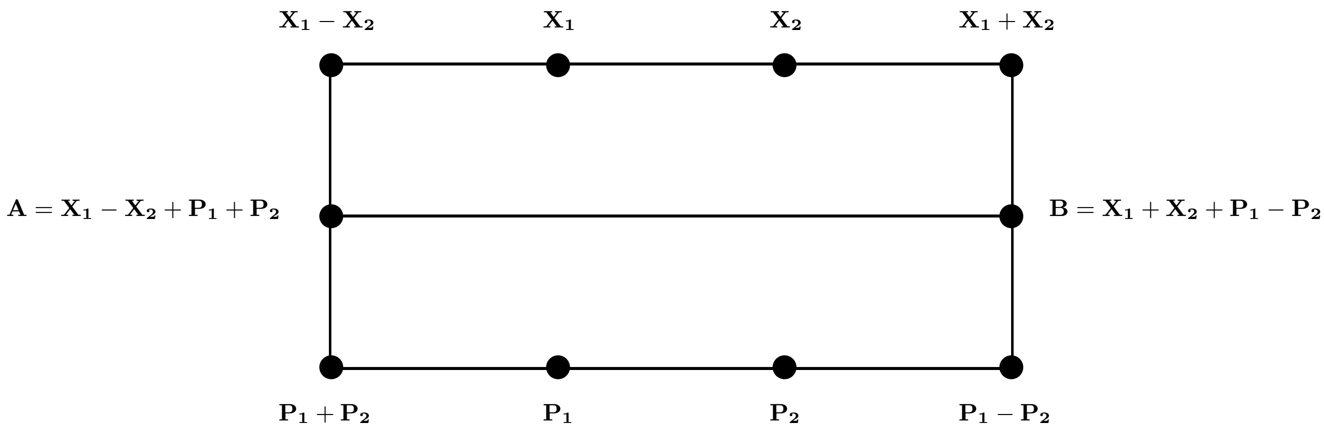


FIG. 1: Set of operators used to show that, although X and P do not commute, their putative values are additive. Notice that all operators joined by a straight line commute.

For this, let us consider an ensemble of systems described by quantum mechanics by a Hilbert space element $\psi$. Let $\{\varphi_x\}$ , $\{\phi_p\}$ , $\{\eta_s\}$ be the eigenvectors of the operators $X, P, S$ corresponding to the eigenvalues $x, p, s$. According to quantum mechanics, these three observables will have the probability distribution functions

$$\rho(x) = |\langle \varphi_x, \psi \rangle|^2 \ , \tag{4}$$

$$\varpi(p) = |\langle \phi_p, \psi \rangle|^2 \ , \tag{5}$$

$$\sigma(s) = |\langle \eta_s, \psi \rangle|^2 \ . \tag{6}$$

The assumptions made are the usual ones when we deal with position and momentum observables. However in order to be more rigourous we should state that the operators $X, P$ and $S$ are unbound (this follows from their commutation relations) and they have no eigenvectors *in* the Hilbert space. The solution to this problem is to define a *Rigged* Hilbert space (known as a Gel'fand triplet in mathematics) that contains the sets of generalized eigenvectors $\{\varphi_x\}$ , $\{\phi_p\}$ , $\{\eta_s\}$ that can be used as basses in order to expand any Hilbert space element[15]. The modulus squared of the expansion coefficients are interpreted in quantum mechanics as the probability distributions given in the equations above. As an example, the generalized eigenvectors in the coordinate representation, where $X = x$ and $P = -i\partial_x$, are given by

$$\varphi_{x_0}(x) = \delta(x - x_0) \ , \tag{7}$$

$$\phi_p(x) = \frac{1}{\sqrt{2\pi}} \exp(ipx) \ , \tag{8}$$

$$\eta_s(x) = \frac{i^{1/4}}{\sqrt{2\pi}} \exp\left(-i\left(\frac{(x-s)^2}{2} - \frac{s^2}{4}\right)\right) \ . \tag{9}$$

One can easily check that they satisfy their corresponding eigenvalue equations and that they are "delta function" normalized. We will not use these functions but we present them just in order to clarify that the probability distributions in Eqs.(4) to (6) are mathematically well defined.

In the ensemble of systems, the putative values of position and momentum $\overline{X}$ and $\overline{P}$ are distributed according to $\rho(x)$ and $\varpi(p)$, then the addition of these two random variables $\overline{S} = \overline{X+P} = \overline{X} + \overline{P}$ is distributed, according to the theory of random variables, by the convolution

$$\begin{aligned} \sigma_{pv}(s) &= \int dx\ \rho(x)\ \varpi(s-x) \\ &= \int dx \int dp\ \rho(x)\ \varpi(p)\ \delta\left(s-(x+p)\right)\ . \end{aligned} \tag{10}$$

Now we will see that this putative value prediction for the distribution is different from the quantum mechanical prediction in Eq.(6), that can be written as

$$\sigma(s) = \int dx \int dp\ \langle\varphi_x, \psi\rangle\langle\psi, \phi_p\rangle\ \langle\phi_p, \eta_s\rangle\langle\eta_s, \varphi_x\rangle\ , \tag{11}$$

and appears formally quite different from the putative value distribution. The formal difference between these two distributions is such that, presumably, $\sigma_{pv}(s) \neq \sigma(s)$ for all states $\psi$. In particular, it is easy to show that for some states both distributions have different dispersion, that is, $\Delta^2_{pv}(s) \neq \Delta^2_{mq}(s)$. The quantum mechanical prediction is:

$$\begin{aligned} \Delta^2_{mq}(s) &= \langle S^2\rangle - \langle S\rangle^2 = \langle (X+P)^2\rangle - \langle X+P\rangle^2 \\ &= \langle X^2 + P^2 + XP + PX\rangle - \langle X\rangle^2 - \langle P\rangle^2 - 2\langle X\rangle\langle P\rangle \\ &= \Delta^2(x) + \Delta^2(p) + \langle XP + PX\rangle - 2\langle X\rangle\langle P\rangle\ , \end{aligned}$$

and the putative value prediction is

$$\begin{aligned} \Delta^2_{pv}(s) &= \int ds\ s^2 \sigma_{pv}(s) - \left(\int ds\ s\ \sigma_{pv}(s)\right)^2 \\ &= \int ds\ s^2 \int dx \int dp\ \rho(x)\ \varpi(p)\ \delta\left(s-(x+p)\right) - \left(\int ds\ s \int dx \int dp\ \rho(x)\ \varpi(p)\ \delta\left(s-(x+p)\right)\right)^2 \\ &= \int dx \int dp\ (x+p)^2 \rho(x)\ \varpi(p) - \left(\int dx \int dp\ (x+p)\ \rho(x)\ \varpi(p)\right)^2 \\ &= \int dx \int dp\ (x^2+p^2+2xp)\ \rho(x)\ \varpi(p) - \left(\int dx\ x\ \rho(x) + \int dp\ p\ \varpi(p)\right)^2 \\ &= \Delta^2(x) + \Delta^2(p)\ . \end{aligned}$$

Clearly, at least for any state with non vanishing correlation[16] $\langle XP + PX\rangle - 2\langle X\rangle\langle P\rangle \neq 0$, the assumption of the existence of the putative values is in contradiction with quantum mechanics.

## IV. CONCLUSION

If the predictions of quantum mechanics are correct, then we have proved that position and momentum observables can not be assigned any context independent value. The proof presented here involves elementary observables of one single particle and provides a very simple illustration of the Bell-Kochen-Specker contradiction.

Context *dependent* putative values are not prohibited and all attempts to replace standard quantum mechanics by some form of hidden variables theories must necessarily include the context dependence in the deterministic assignment

of values to the observables. This necessity makes such deterministic theories less appealing. One of the main reasons for developing hidden variables theories was to bring the quantum world closer to the classical expectations but the necessary contextuality goes in the other direction.

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